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On a characterization of positive maps

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Abstract
Drawing on results of Choi, Størmer and Woronowicz, we present a nearly complete characterization of certain important classes of positive maps. In particular, we construct a general class of positive linear maps acting between two matrix algebras $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional Hilbert spaces. It turns out that elements of this class are characterized by operators from the dual cone of the set of all separable states on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Subsequently, the relation between entanglements and positive maps is described. Finally, a new characterization of the cone $\mathcal{B}(\mathcal{H})^+ \otimes \mathcal{B}(\mathcal{K})^+$ is given.

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1. Introduction

While discussing non-relativistic quantum mechanics, one must mention fundamental open problems which are related to open questions of modern mathematics. A good example is the separability problem, i.e. the question whether a state of a composite system does not contain any quantum correlations. On the other hand (cf [6, 10, 12] and references therein) the separability question is strongly linked to the problem of classification and characterization of positive operator maps on C^*-algebras. Although many results concerning classification of positive maps have been obtained [2–5, 9, 13, 14, 16, 18, 23], the complete classification of positive maps still remains an essentially open question. In this paper we present some new results concerning separable states, decomposable maps and transposable states.

The presentation of these new results and their proofs require some former results which are scattered across the mathematical literature. Thus, they are not very easily accessible to the physics audience. Consequently, we present a survey of the current state of knowledge concerning the classification of positive maps together with our new results on that topic. We also include somewhat simplified proofs of the already established results. Finally, we present a classification which should be very useful for a study of the separability problem in the context of quantum information theory. To illustrate this point we will look more closely at the relation between nonseparable states and the Peres–Horodecki approach.

The paper is organized as follows. In section 2 some basic definitions are recalled. In section 3 we use the construction of Choi [2] for classification of positive maps and decomposable maps, while in section 4 we are concerned with characterization of separable states. Finally, section 5 is devoted to characterization of the operators in $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ that belong to cone $\mathcal{B}(\mathcal{H})^+ \otimes \mathcal{B}(\mathcal{K})^+$, provided that $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional Hilbert spaces.

2. Definitions and notations

For any C^*-algebra $\mathcal{A}$ by $\mathcal{A}^+$ we denote the set of all positive elements of $\mathcal{A}$. If $\mathcal{A}$ is a unital C^*-algebra then a *state* on $\mathcal{A}$ is a linear functional $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ such that $\phi(A) \geqslant 0$ for every $A \in \mathcal{A}^+$ and $\phi(\mathbb{I}) = 1$, where $\mathbb{I}$ is the unit of $\mathcal{A}$. The set of all states on $\mathcal{A}$ is denoted by $\mathcal{S}_{\mathcal{A}}$. For any $\mathcal{T} \subset \mathcal{S}_{\mathcal{A}}$ we define the *dual cone*

$$\mathcal{T}^{\mathrm{d}} = \{A \in \mathcal{A} : \phi(A) \geqslant 0 \text{ for every } \phi \in \mathcal{T}\}.$$

It is easy to check that the definition of a state implies $\mathcal{A}^+ \subset \mathcal{T}^{\mathrm{d}}$ for every $\mathcal{T} \subset \mathcal{S}_{\mathcal{A}}$. We say that the family $\mathcal{T}$ determines the order of $\mathcal{A}$ when $\mathcal{T}^{\mathrm{d}} = \mathcal{A}^+$.

A linear map $\Psi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ between C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is called *positive* if $\Psi(\mathcal{A}_1^+) \subset \mathcal{A}_2^+$. For $k \in \mathbb{N}$ we consider a map $\Psi_k : M_k \otimes \mathcal{A}_1 \longrightarrow M_k \otimes \mathcal{A}_2$ where M_k denotes the algebra of $k \times k$ matrices with complex entries and $\Psi_k = \mathrm{id}_{M_k} \otimes \Psi$. We say that Ψ is *k-positive* if the map Ψ_k is positive. The map Ψ is said to be *completely positive* when Ψ is k-positive for every $k \in \mathbb{N}$.

For any Hilbert space $\mathcal{L}$ by $\mathcal{B}(\mathcal{L})$ we denote the C^*-algebra of all bounded linear operators acting on $\mathcal{L}$. Let us recall that for a finite-dimensional $\mathcal{L}$ every state ϕ on $\mathcal{B}(\mathcal{L})$ has the form of $\phi(A) = \mathrm{Tr}(\varrho A)$, where ϱ is a uniquely determined *density matrix*, i.e. an element of $\mathcal{B}(\mathcal{L})^+$ such that $\mathrm{Tr}\,\varrho = 1$.

Throughout our paper $\mathcal{H}$ and $\mathcal{K}$ will be fixed finite-dimensional Hilbert spaces. We also fix orthonormal bases $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^m$ of the spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, where $n = \dim\mathcal{H}$, $m = \dim\mathcal{K}$. For simplicity we write $\mathcal{S}, \mathcal{S}_{\mathcal{H}}, \mathcal{S}_{\mathcal{K}}$ instead of $\mathcal{S}_{\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathcal{K})}, \mathcal{S}_{\mathcal{B}(\mathcal{H})}, \mathcal{S}_{\mathcal{B}(\mathcal{K})}$, respectively. By $\tau_{\mathcal{H}}$, $\tau_{\mathcal{K}}$, $\tau_{\mathcal{H}\otimes\mathcal{K}}$ we denote transposition maps on $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$, respectively, associated with bases $\{e_i\}$, $\{f_j\}$, $\{e_i \otimes f_j\}$, respectively. Let us note that for every finite-dimensional Hilbert space $\mathcal{L}$ the transposition $\tau_{\mathcal{L}} : \mathcal{B}(\mathcal{L}) \longrightarrow \mathcal{B}(\mathcal{L})$ is a positive map but not completely positive (in fact it is not even 2-positive).

A positive map $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ is called *decomposable* if there are completely positive maps $\Psi_1, \Psi_2 : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ such that $\Psi = \Psi_1 + \Psi_2 \circ \tau_{\mathcal{H}}$. Let $\mathcal{P}$, $\mathcal{P}_{\mathrm{C}}$ and $\mathcal{P}_{\mathrm{D}}$ denote the set of all positive, completely positive and decomposable maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$, respectively. Note that

$$\mathcal{P}_{\mathrm{C}} \subset \mathcal{P}_{\mathrm{D}} \subset \mathcal{P} \tag{2.1}$$

(see also [1]).

A state $\varphi \in \mathcal{S}$ is said to be *separable* if it can be approximated by states of the form

$$\varphi = \sum_{n=1}^{N} a_n \varphi_n^{\mathcal{H}} \otimes \varphi_n^{\mathcal{K}}$$

where $N \in \mathbb{N}$, $\varphi_n^{\mathcal{H}} \in \mathcal{S}_{\mathcal{H}}$, $\varphi_n^{\mathcal{K}} \in \mathcal{S}_{\mathcal{K}}$ for $n = 1, 2, \ldots, N$, a_n are positive numbers such that $\sum_{n=1}^{N} a_n = 1$, and the state $\varphi_n^{\mathcal{H}} \otimes \varphi_n^{\mathcal{K}}$ is defined as $\varphi_n^{\mathcal{H}} \otimes \varphi_n^{\mathcal{K}}(A \otimes B) = \varphi_n^{\mathcal{H}}(A)\varphi_n^{\mathcal{K}}(B)$ for

$A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$. The set of all separable states on the algebra $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ is denoted by $\mathcal{S}_{\text{sep}}$. A state which is not in $\mathcal{S}_{\text{sep}}$ is called *entangled* or *nonseparable*.

Finally, let us define the family of *transposable* states on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$

$$\mathcal{S}_\tau = \{\varphi \in \mathcal{S} : \varphi \circ (\text{id}_{\mathcal{B}(\mathcal{H})} \otimes \tau_{\mathcal{K}}) \in \mathcal{S}\}.$$

Note that due to the positivity of the transposition $\tau_{\mathcal{K}}$ every separable state φ is transposable, so

$$\mathcal{S}_{\text{sep}} \subset \mathcal{S}_\tau \subset \mathcal{S}. \tag{2.2}$$

In the next sections we describe relations between (2.1) and (2.2).

To conclude, we should remark that the application of the remarkable theorems of Tomiyama [20] and Kadison [7] leads to the following result.

Theorem 2.1. *The family $\mathcal{S}_{\text{sep}}$ of separable states on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ is *-weakly dense in $\mathcal{S}$ if and only if* $\dim \mathcal{H} = 1$ *or* $\dim \mathcal{K} = 1$.

3. Construction of positive maps

Now we want to present a general construction of a linear positive map $S : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$. In the sequel we assume that both $\mathcal{H}$ and $\mathcal{K}$ have dimension greater than 1.

For any element $x \in \mathcal{H}$ we define the linear operator $V_x : \mathcal{K} \longrightarrow \mathcal{H} \otimes \mathcal{K}$ by $V_x z = x \otimes z$ for $z \in \mathcal{K}$. By $E_{x,y}$ where $x, y \in \mathcal{H}$ we denote the one-dimensional operator on $\mathcal{H}$ defined by $E_{x,y}u = \langle y, u\rangle x$ for $u \in \mathcal{H}$. For simplicity reasons, if $\{e_i\}_{i=1}^n$ is a basis of $\mathcal{H}$, we write V_i and E_{ij} instead of V_{e_i} and E_{e_i,e_j} for any $i, j = 1, 2, \ldots, n$.

Let us start with the observation that for any $H \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ we have

$$\begin{aligned}\langle z \otimes v, Hx \otimes y\rangle &= \sum_{ij} \overline{\langle e_i, z\rangle}\langle e_j, x\rangle\langle e_i \otimes v, He_j \otimes y\rangle = \sum_{ij}\langle z, E_{ij}x\rangle\langle v, V_i^* H V_j y\rangle \\ &= \left\langle z \otimes v, \left(\sum_{ij} E_{ij} \otimes V_i^* H V_j\right) x \otimes y\right\rangle\end{aligned}$$

where $x, z \in \mathcal{H}$ and $y, v \in \mathcal{K}$. Thus, we have the following decomposition of H:

$$H = \sum_{i,j=1}^{n} E_{ij} \otimes V_i^* H V_j.$$

This suggests that for a fixed H one can define the map $S_H : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$

$$S_H(E_{x,y}) = V_x^* H V_y \tag{3.1}$$

where $x, y \in \mathcal{H}$. The correspondence between H and S_H was observed by Choi [2]. The purpose of this section is to describe properties of the map S_H.

Recall that for two matrices $A = [a_{ij}]_{i,j=1,2,\ldots,n}$ and $B = [b_{ij}]_{i,j=1,2,\ldots,n}$ one can define the Hadamard product $A * B = [a_{ij}b_{ij}]_{i,j=1,2,\ldots,n}$. We will need the following lemma.

Lemma 3.1 ([8], Exercise 2.8.41). *If both matrices A and B are positive definite then their Hadamard product $A * B$ is also positive definite.*

The main property of the map S_H is described by the following proposition.

Proposition 3.2. *If $H^* = H$ and $H \in \mathcal{S}_{\text{sep}}^{\text{d}}$ then S_H is a positive map. Moreover, for any positive map $S : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ there exists uniquely determined selfadjoint operator $H \in \mathcal{S}_{\text{sep}}^{\text{d}}$ such that $S = S_H$.*

Proof. To prove positivity of S_H, observe first that it is invariant with respect to the *-operation:

$$S_H(E_{x,y}^*) = S_H(E_{y,x}) = V_y^* H V_x = (V_x^* H V_y)^* = S_H(E_{x,y})^*.$$

Secondly, it is enough to prove that S_H maps any one-dimensional projector $E_{x,x}$, where $\|x\| = 1$, into a positive operator. To this end we note

$$\langle y, S(E_{x,x})y\rangle = \langle y, V_x^* H V_x y\rangle = \langle V_x y, H V_x y\rangle = \langle x \otimes y, Hx \otimes y\rangle \geqslant 0$$

where the last inequality follows from the fact that $H \in \mathcal{S}_{\rm sep}^{\rm d}$.

Suppose that S is any positive map. Define

$$H = (\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes S)\left(\sum_{k,l=1}^{n} E_{kl} \otimes E_{kl}\right). \tag{3.2}$$

The positivity assumption and the fact that $\sum_{kl} E_{kl} \otimes E_{kl}$ is selfadjoint imply that H is selfadjoint. In order to prove that $H \in \mathcal{S}_{\rm sep}^{\rm d}$, one should show that

$$\varphi^{\mathcal{H}} \otimes \varphi^{\mathcal{K}}(H) \geqslant 0 \tag{3.3}$$

for any $\varphi^{\mathcal{H}} \in \mathcal{S}_{\mathcal{H}}$ and $\varphi^{\mathcal{K}} \in \mathcal{S}_{\mathcal{K}}$. To this end we observe that

$$\varphi^{\mathcal{H}} \otimes \varphi^{\mathcal{K}}(H) = \sum_{kl} \varphi^{\mathcal{H}}(E_{kl})\varphi^{\mathcal{K}}(S(E_{kl})). \tag{3.4}$$

Recall that for any state $\phi \in \mathcal{S}_{\mathcal{H}}$ the matrix $[\phi(E_{kl})]_{k,l=1,2,\ldots,n}$ is positive definite. Thus, both matrices $[\varphi^{\mathcal{H}}(E_{kl})]$ and $[\varphi^{\mathcal{K}}(S(E_{kl}))]$ are positive definite. The right-hand side of equality (3.4) is the sum of all entries of the Hadamard product $[\varphi^{\mathcal{H}}(E_{kl})]*[\varphi^{\mathcal{K}}(S(E_{kl}))]$ which is also positive (cf lemma 3.1). The sum of entries of a positive definite matrix is positive, so (3.3) is proved. To prove that $S_H = S$, it is enough to show that $S_H(E_{ij}) = S(E_{ij})$ for any $i, j = 1, 2, \ldots, n$. But, for $y, w \in \mathcal{K}$ we have

$$\begin{aligned}\langle y, S_H(E_{ij})w\rangle = \langle y, V_i^* H V_j w\rangle &= \left\langle e_i \otimes y, \left(\sum_{kl} E_{kl} \otimes S(E_{kl})\right) e_j \otimes w\right\rangle \\ &= \sum_{kl} \langle e_i, E_{kl} e_j\rangle \langle y, S(E_{kl})w\rangle = \langle y, S(E_{ij})w\rangle.\end{aligned}$$

Thus, the proposition is proved. □

The next proposition characterizes the case when S_H is a completely positive map. Following Choi [2], we have

Proposition 3.3 ([2]). *S_H is a completely positive map if and only if H is a positive operator.*

Corollary 3.4. *If* $\dim \mathcal{H} \geqslant 2$ *and* $\dim \mathcal{K} \geqslant 2$ *then there exists H such that S_H is a positive but not completely positive map.*

Proof. In order to prove this statement one should prove that there exists a selfadjoint operator $H \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that

(i) $\varphi(H) \geqslant 0$ for all $\varphi \in \mathcal{S}_{\rm sep}$;
(ii) $H \notin \mathcal{B}(\mathcal{H} \otimes \mathcal{K})^+$.

But, from the theorem of Tomiyama mentioned in the previous section we get that $\mathcal{S}_{\rm sep}$ does not determine the order of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, so $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})^+$ is a proper subset of $\mathcal{S}_{\rm sep}^{\rm d}$. Any element H of $\mathcal{S}_{\rm sep}^{\rm d} \setminus \mathcal{B}(\mathcal{H} \otimes \mathcal{K})^+$ satisfies both conditions. □

The next proposition describes the properties of positive decomposable maps. It will be done by means of the family of transposable states. We will need the following lemma.

Lemma 3.5. *Let $k \in \mathbb{N}$ and $A \in M_k \otimes \mathcal{B}(\mathcal{H})$. Suppose that both A and $(\tau_{M_k} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})})(A)$ are positive in $M_k \otimes \mathcal{B}(\mathcal{H})$. Then for every vector $x_1, x_2, \ldots, x_k \in \mathcal{K}$ the map $\psi : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \longrightarrow \mathbb{C}$ defined as*

$$\psi(C) = \sum_{i,j=1}^{k} \sum_{p,r=1}^{n} \langle h_i \otimes e_p, Ah_j \otimes e_r \rangle \langle e_p \otimes x_i, Ce_r \otimes x_j \rangle \qquad C \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$$

is a positive functional on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that $\varphi \circ (\tau_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})})$ is also positive.

Proof. First of all note that for every state $\varphi \in \mathcal{S}$ we have the following equivalence:

$$\varphi \in \mathcal{S}_\tau \Longleftrightarrow \varphi \circ (\tau_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})}) \in \mathcal{S}.$$

Observe that

$$\begin{aligned}\psi(C) &= \sum_{i,j,p,r} \langle h_i \otimes e_p \otimes e_p \otimes x_i, (A \otimes C) h_j \otimes e_r \otimes e_r \otimes x_j \rangle \\ &= \left\langle \sum_{i,p} h_i \otimes e_p \otimes e_p \otimes x_i, (A \otimes C) \sum_{i,p} h_i \otimes e_p \otimes e_p \otimes x_i \right\rangle.\end{aligned}$$

If C is positive then $A \otimes C$ is positive in the algebra $M_k \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$, so $\psi(C) \geqslant 0$. On the other hand,

$$\begin{aligned}\psi(\tau_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})})(C) &= \sum_{i,j,p,r} \langle h_i \otimes e_p, Ah_j \otimes e_r \rangle \langle e_r \otimes x_i, Ce_p \otimes x_j \rangle \\ &= \sum_{i,j,p,r} \langle h_i \otimes e_r, (\mathrm{id}_{M_k} \otimes \tau_{\mathcal{H}})(A) h_j \otimes e_p \rangle \langle e_r \otimes x_i, Ce_p \otimes x_j \rangle \\ &= \left\langle \sum_{i,r} h_i \otimes e_r \otimes e_r \otimes x_i, [(\mathrm{id}_{M_k} \otimes \tau_{\mathcal{H}})(A) \otimes C] \sum_{i,r} h_i \otimes e_r \otimes e_r \otimes x_i \right\rangle.\end{aligned}$$

The positivity of $(\tau_{M_k} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})})(A)$ implies the positivity of $(\mathrm{id}_{M_k} \otimes \tau_{\mathcal{H}})(A)$, so by the above arguments, if C is positive then $\psi(\tau_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})})(C) \geqslant 0$. □

Proposition 3.6. *For any selfadjoint operator H the map S_H is decomposable if and only if $H \in \mathcal{S}_\tau^{\mathrm{d}}$.*

Proof. Suppose that $S_H = S_1 + S_2 \circ \tau_{\mathcal{H}}$, where S_1, S_2 are completely positive. Then $H = H_1 + (\tau_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})})(H_2)$ where H_1, H_2 are positive operators such that $S_i = S_{H_i}$, $i = 1, 2$. Let $\varphi \in \mathcal{S}_\tau$. Hence,

$$\varphi(H) = \varphi(H_1) + \varphi(\tau_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})})(H_2) \geqslant 0$$

because both φ and $\varphi(\tau_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{K})})$ are positive functionals.

Conversely, let $H \in \mathcal{S}_\tau^{\mathrm{d}}$. Suppose that $k \in \mathbb{N}$ and $A = [A_{ij}]_{i,j=1,2,\ldots,k} \in M_k \otimes \mathcal{B}(\mathcal{H})$ is such that both A and $(\tau_{M_k} \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})})(A)$ are positive in $M_k \otimes \mathcal{B}(\mathcal{H})$. From the theorem of Størmer ([15], see also [13]) it is enough to show that $(\mathrm{id}_{M_k} \otimes S_H)(A)$ is a positive element in $M_k \otimes \mathcal{B}(\mathcal{K}) \simeq \mathcal{B}(\mathbb{C}^k \otimes \mathcal{K})$. To this end let us fix an element $h \in \mathbb{C}^k \otimes \mathcal{K}$. Let $h = \sum_{s=1}^{k} h_s \otimes x_s$. Then

$$\begin{aligned}\langle h, (\mathrm{id}_{M_k} \otimes S_H)(A) h \rangle &= \sum_{s,t} \sum_{i,j} \sum_{p,r} \langle e_p, A_{ij} e_r \rangle \langle h_s \otimes x_s, (F_{ij} \otimes V_p^* H V_r) h_t \otimes x_t \rangle \\ &= \sum_{s,t} \sum_{i,j} \sum_{p,r} \langle e_p, A_{ij} e_r \rangle \langle h_s, F_{ij} h_t \rangle \langle e_p \otimes x_s, He_r \otimes x_t \rangle \\ &= \sum_{i,j} \sum_{p,r} \langle e_p, A_{ij} e_r \rangle \langle e_p \otimes x_i, He_r \otimes x_j \rangle\end{aligned}$$

where F_{ij} are matrix units in M_k. The last expression is non-negative by lemma 3.5. □

Remark 3.7. In [5, 9, 13, 17] it was shown that, in general, $\mathcal{S}_\tau^{\mathrm{d}}$ is a proper subset of $\mathcal{S}_{\mathrm{sep}}^{\mathrm{d}}$.

So far we do not know any general efficient characterization of states from $\mathcal{S}_\tau$. However, we will formulate some sufficient conditions for a state to be in $\mathcal{S}_\tau$.

Proposition 3.8. *Suppose that ϱ is a density matrix of a state φ on $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$. Let*

$$\varrho = \sum_{i,j=1}^{n} E_{ij}\otimes\varrho_{ij}$$

where $\varrho_{ij}\in\mathcal{B}(\mathcal{K})$ for $i, j = 1, 2, \ldots, n$. If the C^*-algebra $\mathcal{B}$ generated by elements ϱ_{ij} is Abelian, then $\varphi\in\mathcal{S}_\tau$.

Proof. It is enough to show that $(\mathrm{id}_{\mathcal{B}(\mathcal{H})}\otimes\tau_{\mathcal{K}})(\varrho)$ is a positive operator. This is equivalent to the statement that $(\tau_{\mathcal{H}}\otimes\mathrm{id}_{\mathcal{B}(\mathcal{K})})(\varrho)$ is positive. The theorem of Tomiyama [21] asserts that for any C^*-algebra $\mathcal{A}$ the map $(\tau\otimes\mathrm{id}) : M_n\otimes A \longrightarrow M_n\otimes\mathcal{A}$ is positive if and only if A is Abelian. Hence, as $\mathcal{B}$ is Abelian and $\rho\in\mathcal{B}(\mathcal{H})\otimes\mathcal{B}\simeq M_n\otimes\mathcal{B}$, the proposition follows. □

Before the next propositions, whose principal significance is that they allow one to write (or verify) concrete examples of transposable states, we need to make the following remark.

Remark 3.9. Suppose that $\varrho = [\varrho_{ij}]$ is an operator on $\mathcal{H}\otimes\mathcal{K}$. As a result of the Choi–Robertson lemma [3, 13] one has: $[\varrho_{ij}]$ is positive if and only if the matrix $[\tilde{\varrho_{ij}}]_{i,j=1,2,\ldots,n-1}$ is positive, where $\tilde{\varrho_{ij}} = \varrho_{ij} - \varrho_{in}\varrho_{nn}^{-1}\varrho_{nj}$ for $i, j = 1, 2, \ldots, n-1$. Hence, replacing ϱ_{nn} by $\varrho_{nn} + \varepsilon\mathbb{I}$ if necessary we may suppose that ϱ_{nn} is invertible and then by an application of the above lemma we can restrict ourselves to the case of two-dimensional space $\mathcal{K}$.

Proposition 3.10. *Let* $\dim\mathcal{H} = 2$ *and let* $\varrho = [\varrho_{ij}]_{i,j=1,2}$ *($\varrho_{ij}\in\mathcal{B}(\mathcal{K})$ as in the above proposition) be a density matrix on the space $\mathcal{H}\otimes\mathcal{K}$. Assume that there exists a vector $f\in\mathcal{H}$ and a selfadjoint operator A on $\mathcal{H}$ with a property*

$$\langle f\otimes y, \{A\otimes\mathbb{I}, \varrho\} f\otimes y\rangle = 0$$

for any $y\in\mathcal{K}$, where $\{\cdot,\cdot\}$ stands for the anticommutator. Then $\varrho^\tau = [\varrho_{ji}]_{i,j=1,2}$ is also positive.

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis in $\mathcal{H}$. Define

$$\alpha_1 = \langle f, e_1\rangle \qquad \alpha_2 = \langle f, e_2\rangle$$
$$\alpha_3 = \langle Af, e_1\rangle \qquad \alpha_4 = \langle Af, e_2\rangle.$$

From the assumption we have

$$\langle Af\otimes y, \varrho\, f\otimes y\rangle + \langle\varrho\, f\otimes y, Af\otimes y\rangle = 0$$

for every $y\in\mathcal{K}$. Simple calculations lead to

$$\langle y, [(\overline{\alpha_1}\alpha_3 + \alpha_1\overline{\alpha_3})\varrho_{11} + (\overline{\alpha_2}\alpha_3 + \alpha_1\overline{\alpha_4})\varrho_{12} + (\overline{\alpha_1}\alpha_4 + \alpha_2\overline{\alpha_3})\varrho_{21} + (\overline{\alpha_2}\alpha_4 + \alpha_2\overline{\alpha_4})\varrho_{22}]y\rangle = 0.$$

Hence the system $\{\varrho_{11}, \varrho_{12}, \varrho_{21}, \varrho_{22}\}$ is linearly dependent. According to the Choi theorem [3], the matrix $[\varrho_{ji}]_{i,j=1,2}$ is positive. □

Bearing in mind that an element of $M_k\otimes\mathcal{B}(\mathcal{K})$ is positive if and only if it is a sum of matrices of the form $[\varrho_i^*\varrho_j]_{i,j=1,\ldots,k}$, $\varrho_1, \ldots, \varrho_k\in\mathcal{B}(\mathcal{K})$, we can formulate the following proposition.

Proposition 3.11. *Suppose that* $\dim\mathcal{H} = 2$, *$[\varrho_{ij}]_{i,j=1,2}$ is a positive operator on $\mathcal{H}\otimes\mathcal{K}$ and $\varrho_{ij} = \varrho_i^*\varrho_j$ for some operators $\varrho_1, \varrho_2\in\mathcal{B}(\mathcal{K})$ such that $|\varrho_2| = (\varrho_2^*\varrho_2)^{\frac{1}{2}}$ is an invertible operator. If $b = |\varrho_2|^{-1}\varrho_1^*\varrho_2|\varrho_2|^{-1}$ is a hypernormal operator, then the state determined by $[\varrho_{ij}]$ is transposable.*

Proof. By the Choi–Robertson lemma (cf remark 3.9) the positivity of the matrix $\varrho = [\varrho_i^* \varrho_j]_{i,j=1,2}$ is equivalent to the following inequality:

$$\varrho_1^*\varrho_1 \geqslant \varrho_1^*\varrho_2(\varrho_2^*\varrho_2)^{-1}\varrho_2^*\varrho_1.$$

Consequently, we get

$$|\varrho_2|^{-1}|\varrho_1|^2|\varrho_2|^{-1} \geqslant (|\varrho_2|^{-1}\varrho_1^*\varrho_2|\varrho_2|^{-1})(|\varrho_2|^{-1}\varrho_2^*\varrho_1|\varrho_2|^{-1}) = bb^*.$$

From the hypernormality of b we have $b^*b \leqslant bb^*$. Hence,

$$|\varrho_2|^{-1}|\varrho_1|^2|\varrho_2|^{-1} \geqslant b^*b.$$

This inequality is equivalent to the positivity of the transposed matrix $\varrho = [\varrho_j^*\varrho_i]_{i,j=1,2}$. □

We summarize this section with the following theorem.

Theorem 3.12. *Define a function* Ψ *which to every* $H \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ *assigns the map* S_H *defined by (4.1). Then* $\Psi : \mathcal{S}^{\mathrm{d}}_{\mathrm{sep}} \longrightarrow \mathcal{P}$ *is a bijective convex map. Moreover,*

$$\Psi(\mathcal{S}^{\mathrm{d}}) = \mathcal{P}_{\mathrm{C}} \qquad \Psi(\mathcal{S}^{\mathrm{d}}_{\tau}) = \mathcal{P}_{\mathrm{D}}.$$

4. Characterization of separable states

As it was mentioned in the introduction, the general structure of positive maps is closely related to the problem of separable states [6, 12]. In this section, having already described general classification of positive maps, we want to clarify the relation between nonseparable states and the Peres–Horodecki approach. To this end we will need a well defined measure of entanglement $\mathcal{E}$, (cf [11]).

Let us consider C^*-algebras $\mathcal{A}$, $\mathcal{B}$, and let ω be a state on $\mathcal{A}\otimes\mathcal{B}$. Define a map $r : \mathcal{S}(\mathcal{A}\otimes\mathcal{B}) \longrightarrow \mathcal{S}(\mathcal{A})$ by the following formula:

$$r\omega(a) = \omega(a\otimes \mathbb{I}) \tag{4.1}$$

where $a \in \mathcal{A}$, $\mathbb{I}$ is the unit of $\mathcal{B}$.

First, we prove the following proposition.

Proposition 4.1. *If* $r\omega$ *is a pure state on* $\mathcal{A}$ *then* ω *is a product state, i.e.*

$$\omega(x\otimes y) = \omega^{\mathcal{A}}(x)\omega^{\mathcal{B}}(y) \tag{4.2}$$

where $\omega^{\mathcal{A}} \in \mathcal{S}(\mathcal{A})$, $\omega^{\mathcal{B}} \in \mathcal{S}(\mathcal{B})$ *are defined as*

$$\omega^{\mathcal{A}}(x) = \omega(x\otimes \mathbb{I}) \qquad \omega^{\mathcal{B}}(y) = \omega(\mathbb{I}\otimes y).$$

Moreover, if we assume that ω *is a pure state, then the converse implication is also valid.*

Proof. We present a slight modification of argument given by Takesaki (cf lemma 4.11 in [19]). Assume y is a positive element in the unit ball of $\mathcal{B}$. If $\omega(\mathbb{I}\otimes y) = 0$, then from the Cauchy–Schwarz inequality for states, we have

$$\begin{aligned}|\omega(x\otimes y)|^2 &= \big|\omega\big(\big(x\otimes y^{\frac12}\big)\big(\mathbb{I}\otimes y^{\frac12}\big)\big)\big|^2\\ &\leqslant \omega\big(\big(x\otimes y^{\frac12}\big)\big(x\otimes y^{\frac12}\big)^*\big)\omega\big(\big(\mathbb{I}\otimes y^{\frac12}\big)\big(\mathbb{I}\otimes y^{\frac12}\big)^*\big)\\ &= \omega\big(xx^*\otimes y\big)\omega(\mathbb{I}\otimes y) = 0.\end{aligned}$$

So, (4.2) holds. If $\omega(\mathbb{I}\otimes y) = 1$ then we apply the above argument to $\mathbb{I}\otimes(\mathbb{I}-y)$ instead of $\mathbb{I}\otimes y$, to obtain $\omega(x\otimes(\mathbb{I}-y)) = 0$. This can be rewritten as $\omega(x\otimes\mathbb{I}) = \omega(x\otimes y)$. Consequently, we get

$$\omega(x\otimes y) = \omega^{\mathcal{A}}(x)\cdot 1 = \omega^{\mathcal{A}}(x)\omega^{\mathcal{B}}(y).$$

Suppose now that $0 < \omega(\mathbb{I} \otimes y) < 1$. Let $\omega_1, \omega_2 \in \mathcal{S}(\mathcal{A})$ be defined as

$$\omega_1(x) = \frac{1}{\omega(\mathbb{I} \otimes y)}\omega(x \otimes y) \qquad \omega_2(x) = \frac{1}{1 - \omega(\mathbb{I} \otimes y)}\omega(x \otimes (1 - y))$$

for $x \in \mathcal{A}$. Then, we have

$$\omega^{\mathcal{A}}(x) = \omega(\mathbb{I} \otimes y)\omega_1(x) + (1 - \omega(\mathbb{I} \otimes y))\omega_2(x).$$

Hence, by the assumption of our proposition, we have $\omega_1(x) = \omega_2(x)$, so that

$$\omega(x \otimes y) = \omega(x \otimes \mathbb{I})\omega(\mathbb{I} \otimes y)$$

for $x \in \mathcal{A}$. As every $y \in \mathcal{A}$ is a linear combination of positive elements, one can easily extend the above property on every $y \in \mathcal{A}$.

Assume now that ω is a pure product state on $\mathcal{A} \otimes \mathcal{B}$: $\omega(x \otimes y) = \omega^{\mathcal{A}}(x)\omega^{\mathcal{B}}(y)$. Suppose that $r\omega = \omega^{\mathcal{A}} = \lambda_1\varphi_1 + \lambda_2\varphi_2$ for some states φ_1, φ_2 on $\mathcal{B}(\mathcal{H})$, $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$. Then

$$\omega(x \otimes y) = \lambda_1\varphi_1(x)\omega^{\mathcal{B}}(y) + \lambda_2\varphi_2(x)\omega^{\mathcal{B}}(y) \qquad x \in \mathcal{B}(\mathcal{H}) \quad y \in \mathcal{B}(\mathcal{K}).$$

By the assumption we obtain $\varphi_1(x)\omega^{\mathcal{B}}(y) = \varphi_2(x)\omega^{\mathcal{B}}(y)$ for $x \in \mathcal{B}(\mathcal{H})$, $y \in \mathcal{B}(\mathcal{K})$. Consider this equality for $y = \mathbb{I}$ to derive $\varphi_1 = \varphi_2$. □

Remark 4.2. Lemma 11.3.6 of [8] states that the map $r : \mathcal{S}(\mathcal{A} \otimes \mathcal{B}) \longrightarrow \mathcal{S}(\mathcal{A})$ is surjective.

We will need the von Neumann entropy function $s : [0, 1] \longrightarrow \mathbb{R}$ defined by the formula

$$s(x) = \begin{cases} -x\ln x & \text{for} \quad x \in (0, 1] \\ 0 & \text{for} \quad x = 0. \end{cases}$$

It is easy to show that the function is non-negative. Moreover, $s(x) = 0$ if and only if $x = 0$ or $x = 1$. Now, for every density matrix ϱ on $\mathcal{H}$ we define its von Neumann entropy:

$$S(\varrho) = \operatorname{Tr} s(\varrho).$$

Proposition 4.3. $S(\varrho) = 0$ *if and only if* ϱ *is a one-dimensional ortogonal projector.*

Proof. The proof is straightforward and we leave it to the reader. □

Again, as in section 3, let us put $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\mathcal{B} = \mathcal{B}(\mathcal{K})$ with finite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Furthermore, the density matrix of a state ω will be denoted as ϱ_ω. Let us observe that ω is a pure state if and only if ϱ_ω is a one-dimensional projector.

Let $M_1(\mathcal{S})$ denote the set of all probability Radon measures on $\mathcal{S}$. If $\mu \in M_1(\mathcal{S})$, then its barycentre is defined as $b(\mu) = \int_{\mathcal{S}} \phi \, \mathrm{d}\mu\,(\phi)$. For every state $\omega \in \mathcal{S}$ we define $M_\omega(\mathcal{S}) = \{\mu \in M_1(\mathcal{S}) : b(\mu) = \omega\}$.

Definition 4.4. *Let* $\omega \in \mathcal{S}$. *Then we define*

$$\mathcal{E}(\omega) = \inf_{\mu \in M_\omega(\mathcal{S})} \int_{\mathcal{S}} S(\varrho_{r\phi}) \, \mathrm{d}\mu\,(\phi) \tag{4.3}$$

where, as before, r *denotes the restriction map,* S *stands for the entropy:* $S(\varrho) = \operatorname{Tr} s(\varrho)$ *for any density matrix* ϱ. $\mathcal{E}(\omega)$ *will be called the measure of entanglement of the state* ω.

Proposition 4.5. *If* $\omega \in \mathcal{S}$, *then there exists a measure* $\mu_0 \in M_\omega(\mathcal{S})$ *such that* $\mathcal{E}(\omega) = \int_{\mathcal{S}} S(\varrho_{r\phi}) \, \mathrm{d}\mu_0\,(\phi)$, *i.e. the infimum in formula (4.3) is reached.*

Proof. The set $M_1(\mathcal{S})$ is compact in *-weak topology. Moreover, the map $b : M_1(\mathcal{S}) \longrightarrow \mathcal{S}$, $b(\mu) = \int_{\mathcal{S}} \phi \, \mathrm{d}\mu\,(\phi)$ is continuous, hence $M_\omega(\mathcal{S}) = b^{-1}(\{\omega\})$ is a closed subset of $M_1(\mathcal{S})$, so it is compact. Now, consider the map $M_\omega(\mathcal{S}) \longrightarrow \mathbb{R}$ that assigns $\int_{\mathcal{S}} S(\varrho_{r\phi}) \, \mathrm{d}\mu\,(\phi)$ to every $\mu \in M_\omega(\mathcal{S})$. It is continuous because both $r : \mathcal{S} \longrightarrow \mathcal{S}_{\mathcal{B}(\mathcal{H})}$, $S(\varrho_{r\cdot}) : \mathcal{S}_{\mathcal{B}(\mathcal{H})} \longrightarrow \mathbb{R}$ are continuous maps. The assertion of the proposition is the consequence of the Weierstrass theorem. □

Remark 4.6. As a matter of fact, the measure $\mathcal{E}$ appeared in [22] under the name of the formation of entanglement. We prefer to call $\mathcal{E}$ the measure of entanglement since we are able to establish a nice criterion of separability in our next theorem.

Theorem 4.7. *Let $\omega \in \mathcal{S}$. Then, ω is separable if and only if $\mathcal{E}(\omega) = 0$.*

Proof. Assume $\mathcal{E}(\omega) = 0$. Then, by the above proposition, there exists $\mu_0 \in M_\omega(\mathcal{S})$ such that

$$\int_{\mathcal{S}} S(\varrho_{r\phi}) \, \mathrm{d}\mu_0\,(\phi) = 0.$$

As the map $\phi \mapsto S(\varrho_{r\phi})$ is non-negative and continuous then, by proposition 4.3, $\varrho_{r\phi}$ are one-dimensional projectors and $r\phi$ is a pure state for every ϕ from the support of the measure μ_0. Consequently, every $\phi \in \mathrm{supp}\mu_0$ is a product state (cf proposition 4.1). The measure μ_0 can be approximated by finitely supported measures with their supports contained in supp μ_0. Comparing this fact with the equality $\omega = \int_{\mathcal{S}} \phi \, \mathrm{d}\mu\,(\phi)$, we conclude that ω is separable.

Conversely, assume that ω is a separable state. So,

$$\omega = \lim_N \sum_i \lambda_i^{(N)} \omega_i^{(N)}$$

where $\omega_i^{(N)}$ are product states, and the limit is in the weak sense. Let us notice that $\omega_i^{(N)}$ can be chosen in such a way that $r\omega_i^{(N)}$ are pure states. Denote $\mu_N = \sum_i \lambda_i^{(N)} \delta_{\omega_i^{(N)}}$ where $\delta_{\omega_i^{(N)}}$ are Dirac measures at the point $\omega_i^{(N)}$. Then, the sequence (μ_N) contains a convergent subsequence (μ_{N_k}) because $M_1(\mathcal{S})$ is compact. Let $\mu_0 = \lim_k \mu_{N_k}$. Now we have

$$\omega = \lim_k \int \phi \, \mathrm{d}\mu_{N_k}\,(\phi) = \int \phi \, \mathrm{d}\mu_0\,(\phi)$$

so $\mu_0 \in M_\omega(\mathcal{S})$. Moreover,

$$\int S(\varrho_{r\phi}) \, \mathrm{d}\mu_0\,(\phi) = \lim_k \int S(\varrho_{r\phi}) \, \mathrm{d}\mu_{N_k}\,(\phi) = \lim_k \sum_i \lambda_i^{(N_k)} S(\varrho_{r\omega_i^{(N_k)}}) = 0$$

because $r\omega_i^{(N_k)}$ are pure states. Consequently, $\mathcal{E}(\omega) = 0$. □

Let us recall that partial transposition $\mathrm{id}_{\mathcal{H}} \otimes \tau_{\mathcal{K}} : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ is the main ingredient of the Peres–Horodecki approach. Let us observe that

$$r[\omega \circ (\mathrm{id}_{\mathcal{H}} \otimes \tau_{\mathcal{K}})](A) = \omega \circ (\mathrm{id}_{\mathcal{H}} \otimes \tau_{\mathcal{K}})(A \otimes \mathbb{I}) = \omega(A \otimes \mathbb{I}) = (r\omega)(A)$$

for $A \in \mathcal{B}(\mathcal{H})$. Hence,

$$\mathcal{E}(\omega) = \mathcal{E}(\omega \circ (\mathrm{id}_{\mathcal{H}} \otimes \tau_{\mathcal{K}}))$$

if $\omega \circ (\mathrm{id}_{\mathcal{H}} \otimes \tau_{\mathcal{K}}) \in \mathcal{S}$, i.e. ω is a transposable state. Therefore, the problem of detecting nonseparability is reduced to the characterization of $\mathcal{S}_\tau$. It is an easy observation that

$$r(\omega \circ (\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes S)) = r\omega$$

where $S : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ is a positive, unital map. Therefore, again, the main question is to assure that

$$\omega \circ (\mathrm{id}_{\mathcal{H}} \otimes S) \in \mathcal{S}. \tag{4.4}$$

As the set of states satisfying relation (4.5), for finite-dimensional case, can be identified with $M^+ \otimes M^+$ (see [6]), in the next section we provide a characterization of that cone.

5. Characterization of $M_n(\mathbb{C})^+ \otimes M_m(\mathbb{C})^+$

In this section we characterize elements of the cone $\mathcal{B}(\mathcal{H})^+ \otimes \mathcal{B}(\mathcal{K})^+$. Recall that, in general, it is a proper subset of $(\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))^+$.

Lemma 5.1. *Let P be a one-dimensional projector. Then $P \in \mathcal{B}(\mathcal{H})^+ \otimes \mathcal{B}(\mathcal{K})^+$ if and only if*

$$(B_P B_P^\tau)^2 = B_P B_P^\tau \tag{5.1}$$

where $B_P = [\beta_{ij}]_{i=1,\dots,n;\, j=1,\dots,m}$, $\beta_{ij} = \langle \xi, e_i \otimes f_j \rangle$ for $\xi \in H \otimes K$ such that $\|\xi\| = 1$ and $P\xi = \xi$.

Proof. Let $\omega(a) = \mathrm{Tr}\, Pa$ for $a \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$. Observe that

$$P \in \mathcal{B}(\mathcal{H})^+ \otimes \mathcal{B}(\mathcal{K})^+ \iff P = P_{\mathcal{H}} \otimes P_{\mathcal{K}}$$

for some one-dimensional projectors $P_{\mathcal{H}} \in \mathcal{B}(\mathcal{H})$, $P_{\mathcal{K}} \in \mathcal{B}(\mathcal{K})$. Hence, $P \in \mathcal{B}(\mathcal{H})^+ \otimes \mathcal{B}(\mathcal{K})^+$ if and only if ω is a product state. By the last statement of proposition 4.1 this is equivalent to the fact that $r\omega$ is a pure state on $\mathcal{B}(\mathcal{H})$, hence ϱ is a one-dimensional projector on $\mathcal{B}(\mathcal{H})$, where ϱ is the density matrix of the state $r\omega$.

Firstly, let us observe that $\varrho = \mathrm{Tr}_2\, P$, where Tr_2 is the partial trace $\mathrm{Tr}_2(a \otimes b) = a\, \mathrm{Tr}\, b$, $a \in \mathcal{B}(\mathcal{H})$, $b \in \mathcal{B}(\mathcal{K})$.

Let $\{e_i\}$, $\{f_j\}$ be orthonormal bases in $\mathcal{H}$, $\mathcal{K}$ respectively. Define $U_j : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{K}$ by $U_j x = x \otimes f_j$, $j = 1, 2, \dots, m$. For every $x, y \in H$ we have

$$\langle x, \mathrm{Tr}_2\, Py \rangle = \sum_j \langle x \otimes f_j, Py \otimes f_j \rangle = \left\langle x \sum_j U_j^* P U_j y \right\rangle.$$

So, $\varrho = \mathrm{Tr}_2\, P = \sum_j U_j^* P U_j$ and it is clearly selfadjoint.

Let us examine the conditions leading to the idempotent property of ϱ. To this end we start with derivation of the formula for $U_l U_k^*$. We observe that for any $x \in \mathcal{H}$

$$\langle U_k^* e_i \otimes f_j, x \rangle = \langle e_i \otimes f_j, U_k x \rangle = \langle e_i \otimes f_j, x \otimes f_k \rangle = \langle \delta_{jk} e_i, x \rangle.$$

Hence $U_k^* e_i \otimes f_j = \delta_{jk} e_i$, and

$$U_l U_k^* e_i \otimes f_j = \delta_{jk} e_i \otimes f_l.$$

Suppose $\varrho^2 = \varrho$ and take an arbitrary vector $x \in \mathcal{H}$. Then

$$\begin{aligned}
\varrho^2 x = (\mathrm{Tr}_2\, P)^2 x &= \sum_{jk} U_j^* P U_j U_k^* P U_k x = \sum_{jk} U_j^* P U_j U_k^* P x \otimes f_k \\
&= \sum_{jk} U_j^* P U_j U_k^* \langle \xi, x \otimes f_k \rangle \xi = \sum_{jk} \langle \xi, x \otimes f_k \rangle U_j^* P U_j U_k^* \sum_{pr} \beta_{pr} e_p \otimes f_r \\
&= \sum_{jkpr} \langle \xi, x \otimes f_k \rangle \beta_{pr} U_j^* P U_j U_k^* e_p \otimes f_r = \sum_{jkpr} \langle \xi, x \otimes f_k \rangle \beta_{pr} U_j^* P \delta_{kr} e_p \otimes f_j \\
&= \sum_{jkp} \langle \xi, x \otimes f_k \rangle \beta_{pk} U_j^* P e_p \otimes f_j \\
&= \sum_{jkp} \langle \xi, x \otimes f_k \rangle \beta_{pk} U_j^* \langle \xi, e_p \otimes f_j \rangle \sum_{st} \beta_{st} e_s \otimes f_t \\
&= \sum_{jkp} \sum_{st} \langle \xi, x \otimes f_k \rangle \beta_{pk} \beta_{pj} \beta_{st} U_j^* e_s \otimes f_t = \sum_{jkps} \langle \xi, x \otimes f_k \rangle \beta_{pk} \beta_{pj} \beta_{sj} e_s \\
&= \sum_s \left(\sum_{jkp} \langle \xi, x \otimes f_k \rangle \beta_{pk} \beta_{pj} \beta_{sj} \right) e_s.
\end{aligned}$$

On the other hand

$$\begin{aligned}\varrho x &= \mathrm{Tr}_2\, Px = \sum_k U_k^* P U_k x = \sum_k U_k^* P x \otimes f_k \\ &= \sum_k \langle \xi, x \otimes f_k \rangle U_k^* \sum_{st} \beta_{st} e_s \otimes f_t = \sum_{kst} \langle \xi, x \otimes f_k \rangle \beta_{st} U_k^* e_s \otimes f_t \\ &= \sum_{kst} \langle \xi, x \otimes f_k \rangle \beta_{st} \delta_{kt} e_s = \sum_s \left(\sum_k \langle \xi, x \otimes f_k \rangle \beta_{sk} \right) e_s.\end{aligned}$$

Hence

$$\sum_k \langle \xi, x \otimes f_k \rangle \beta_{sk} = \sum_{jkp} \langle \xi, x \otimes f_k \rangle \beta_{pk} \beta_{pj} \beta_{sj}$$

for every $x \in \mathcal{H}$, $s = 1, \ldots, n$. So,

$$\sum_k \beta_{ik} \beta_{sk} = \sum_{jkp} \beta_{ik} \beta_{pk} \beta_{pj} \beta_{sj}$$

for every $i, s = 1, \ldots, n$. Hence, (5.1) follows. □

As a corollary we get the following theorem.

Theorem 5.2. *A positive operator A on $\mathcal{H} \otimes \mathcal{K}$ belongs to $\mathcal{B}(\mathcal{H})^+ \otimes \mathcal{B}(\mathcal{K})^+$ if and only if there exists a spectral decomposition of A*

$$A = \sum_i \lambda_i P_i$$

where P_i are one-dimensional projectors and $(B_{P_i} B_{P_i}^{\tau})^2 = B_{P_i} B_{P_i}^{\tau}$ for every i, where B_{P_i} were defined in lemma 5.1.

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